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# Hyponormal and strongly hyponormal matrices in inner product spaces

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## ABSTRACT

Complex matrices that are structured with respect to a possibly degenerate indefinite inner product are studied. Based on earlier works on normal matrices, the notions of hyponormal and strongly hyponormal matrices are introduced. A full characterization of such matrices is given and it is shown how those matrices are related to different concepts of normal matrices in degenerate inner product spaces. Finally, the existence of invariant semidefinite subspaces for strongly hyponormal matrices is discussed.

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## 1. Introduction

We consider the space  $\mathbb{C}^n$  equipped with an indefinite inner product  $[\cdot, \cdot]$  that is not necessarily nondegenerate, i.e., there may exist vectors  $x \in \mathbb{C}^n \setminus \{0\}$  such that  $[x, y] = 0$  for all  $y \in \mathbb{C}^n$ . In the case

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of a nondegenerate inner product  $[\cdot, \cdot]$ , the *adjoint* of a matrix  $T$  with respect to  $[\cdot, \cdot]$  is the unique matrix  $T^{[*]}$  satisfying

$$[x, Ty] = [T^{[*]}x, y] \quad \text{for all } x, y \in \mathbb{C}^n. \quad (1)$$

As usual, one defines *H-selfadjoint*, *H-skewadjoint*, *H-unitary*, and *H-normal* matrices, as matrices satisfying

$$T^{[*]} = T, \quad T^{[*]} = -T, \quad T^{[*]} = T^{-1}, \quad \text{and} \quad T^{[*]}T = TT^{[*]}, \quad (2)$$

respectively. Introducing the Gram matrix  $H$  via

$$[x, y] = (Hx, y),$$

where  $(\cdot, \cdot)$  denotes the standard Euclidean scalar product, the adjoint can be expressed as

$$T^{[*]} = H^{-1}T^*H$$

and the identities in (2) reduce to

$$HT = T^*H, \quad T^*H + HT = 0, \quad T^*HT = H, \quad HTH^{-1}T^*H = T^*HT, \quad (3)$$

respectively.

*H*-Selfadjoint, *H*-skewadjoint, *H*-unitary, and *H*-normal matrices have been studied extensively in the literature. Interest is motivated by various applications such as the theory of zones of stability for linear differential equations with periodic coefficients, see [8], the theory of algebraic Riccati equations, see [9], and the linear quadratic optimal control problems as in [17]. A concise overview of the theory of matrices in spaces with an indefinite inner product can be found in [3,4], see also [5] for *H*-normal matrices.

An even more general class of matrices is the set of *H-hyponormal* matrices that are defined by analogy to the well-known class of hyponormal operators in Hilbert spaces via the condition

$$H(T^{[*]}T - TT^{[*]}) \geq 0. \quad (4)$$

For negative definite  $H$ , the set of these matrices equals the set of *H*-normal matrices, but in the case that  $H$  is not definite, the set of *H*-hyponormal matrices is a proper superset of the set of *H*-normal matrices. *H*-hyponormal matrices were studied in detail in [13,14], where, in particular, extension results of invariant semidefinite subspaces to invariant maximal semidefinite subspaces were obtained.

Spaces with a degenerate inner product, i.e., the Gram matrix  $H$  is singular, are less familiar, although this case does appear in applications, e.g., in the theory of operator pencils, cf. [10]. The main problem in this context is that there is no straightforward definition of an *H*-adjoint. Indeed, if  $H$  is noninvertible, the *H*-adjoint of a matrix  $T \in \mathbb{C}^{n \times n}$  need not exist. For example consider

$$H = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \text{and} \quad T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Then a simple calculation shows that there is no matrix  $N \in \mathbb{C}^{2 \times 2}$  such that

$$(Hx, Ty) = [x, Ty] = [Nx, y].$$

In [15], *H*-selfadjoint, *H*-skewadjoint, and *H*-unitary matrices were defined by using the matrix identities from (3). The corresponding equation for *H*-normal matrices, however, requires an inverse of  $H$ . One way to modify this definition is the use of the well-known Moore–Penrose generalized inverse  $H^\dagger$  of  $H$ . In [11], a matrix  $T$  is called *H*-normal if

$$HTH^\dagger T^*H = T^*HT.$$

We will call such matrices *Moore–Penrose H-normal matrices* in this paper.

In [16], a different definition of *H*-normal matrices in degenerate inner product spaces was used which is based on a generalization of the *H*-adjoint  $T^{[*]}$  of a matrix  $T$  for the case of singular  $H$ . This is obtained by dropping the assumption that the *H*-adjoint of a matrix has to be a matrix itself. Instead,

the  $H$ -adjoint  $T^{[*]}$  is understood as a linear relation in  $\mathbb{C}^n$ , i.e., a subspace of  $\mathbb{C}^{2n}$ . Clearly, every matrix  $T \in \mathbb{C}^{n \times n}$  can be interpreted as a linear relation in  $\mathbb{C}^n$  by identification with its graph

$$\Gamma(T) := \left\{ \begin{pmatrix} x \\ Tx \end{pmatrix}, x \in \mathbb{C}^n \right\} \subseteq \mathbb{C}^{2n}.$$

If  $H \in \mathbb{C}^{n \times n}$  is invertible, then  $T^{[*]}$ , defined as in (1), coincides with the linear relation

$$\left\{ \begin{pmatrix} y \\ z \end{pmatrix} \in \mathbb{C}^{2n} : [y, Tx] = [z, x] \text{ for all } x \in \mathbb{C}^n \right\}. \quad (5)$$

Hence, it is natural to define the adjoint  $T^{[*]}$  of  $T$  with respect to some degenerate inner product as the linear relation given in (5).

This approach was used in [16] to generalize the notion of  $H$ -normal matrices to degenerate inner product spaces: a matrix  $T \in \mathbb{C}^{n \times n}$  is called  $H$ -normal if  $TT^{[*]} \subseteq T^{[*]}T$ . It was then shown in [16] that  $H$ -normal matrices  $T$  have the property that the kernel of  $H$  is  $T$ -invariant. This fact allowed the immediate generalization of extension results of invariant semidefinite subspaces from [13,14] to the degenerate case. However, the fact that the kernel of  $H$  is invariant is not needed in order to obtain results on the existence of invariant maximal semidefinite subspaces. Indeed, it was shown in [12] that the kernel of  $H$  need not be invariant for Moore–Penrose  $H$ -normal matrices, but it is always contained in an invariant  $H$ -neutral subspace. This property was used in the proof of the existence of invariant  $H$ -nonnegative subspaces for Moore–Penrose  $H$ -normal matrices in [12].

In this paper, we continue the work started in [16] by generalizing the concept of  $H$ -hyponormality to the case of degenerate inner product spaces. Our aim is to do this in such a way that the obtained set of matrices

- (i) contains the sets of  $H$ -normal and Moore–Penrose  $H$ -normal matrices;
- (ii) equals the set of  $H$ -normal matrices when  $H$  is negative semi-definite;
- (iii) guarantees that the kernel of  $H$  is always contained in an invariant  $H$ -neutral subspace.

The latter condition will allow the generalization of existence results for invariant maximal  $H$ -nonpositive subspaces.

After reviewing some basic results on linear relations in degenerate indefinite inner product spaces in Section 2, we introduce  $H$ -hyponormal matrices in Section 3. It turns out that this rather straightforward generalization of  $H$ -hyponormality is not satisfactory as the resulting matrices are too general. Therefore, the more restrictive concept of strong hyponormality is introduced in Section 4. In Section 5, we investigate the relation of  $H$ -hyponormal and strongly  $H$ -hyponormal matrices to Moore–Penrose  $H$ -normal matrices. In particular, we show that the set of Moore–Penrose  $H$ -normal matrices is a proper subset of the sets of strongly  $H$ -hyponormal and  $H$ -hyponormal matrices. Finally, we give sufficient conditions for the existence of invariant  $H$ -nonpositive subspaces for strongly  $H$ -hyponormal matrices in Section 6.

## 2. Preliminaries

For the remainder of the paper let  $H \in \mathbb{C}^{n \times n}$  be a possibly singular Hermitian matrix and let  $[\cdot, \cdot]$  denote the possibly degenerate inner product given by

$$[x, y] := (x, Hy) \quad \text{for } x, y \in \mathbb{C}^{n \times n}.$$

If  $\mathcal{L} \subset \mathbb{C}^n$  is a subspace, the  $H$ -orthogonal companion of  $\mathcal{L}$  (in  $\mathbb{C}^n$ ) is defined by

$$\mathcal{L}^{[\perp]} := \{x \in \mathbb{C}^n : [x, \ell] = 0 \text{ for all } \ell \in \mathcal{L}\}.$$

The isotropic part of  $\mathcal{L}$  is defined by

$$\mathcal{L}^\circ := \mathcal{L} \cap \mathcal{L}^{[\perp]}.$$

The subspace  $\mathcal{L}$  is called *nondegenerate* (or, more precisely, *H-nondegenerate*) if  $\mathcal{L}^\circ = \{0\}$ . If  $\mathcal{N} \subset \mathbb{C}^{n \times n}$  is a subspace with  $\mathcal{N} \subset \mathcal{L}^{\perp\perp}$  we write  $\mathcal{N}[\perp]\mathcal{L}$ . If, in addition,  $\mathcal{N} \cap \mathcal{L} = \{0\}$ , then by  $\mathcal{N}[\perp]\mathcal{L}$  we denote the *direct H-orthogonal sum* of  $\mathcal{N}$  and  $\mathcal{L}$ .

A vector  $x \in \mathbb{C}^{n \times n}$  is called *H-positive* (*H-negative*, *H-neutral*) if  $[x, x] > 0$  (resp.  $[x, x] < 0$ ,  $[x, x] = 0$ ), and *H-nonnegative* (*H-nonpositive*) if  $x$  is not *H-negative* (resp. not *H-positive*). A subspace  $\mathcal{L} \subset \mathbb{C}^{n \times n}$  is called *H-positive* (*H-negative*, *H-neutral*, *H-nonnegative*, *H-nonpositive*) if all vectors in  $\mathcal{L} \setminus \{0\}$  are *H-positive* (resp. *H-negative*, *H-neutral*, *H-nonnegative*, *H-nonpositive*). Observe that by this definition the zero space  $\{0\}$  is both *H-positive* and *H-negative*. The subspace  $\mathcal{L}$  is called *maximal H-nonpositive* if it is *H-nonpositive* and if there is no nonpositive subspace  $\mathcal{L}' \neq \mathcal{L}$  containing  $\mathcal{L}$ .

For basic facts concerning the geometry in spaces with a degenerate inner product we refer to [1].

## 2.1. Linear relations

The proofs for the propositions and lemmas used in this section can be found, e.g., in [2,7,16]. A *linear relation* in  $\mathbb{C}^n$  is a linear subspace of  $\mathbb{C}^{2n}$ . A matrix  $T \in \mathbb{C}^{n \times n}$  can be interpreted as a linear relation in  $\mathbb{C}^n$  via its graph  $\Gamma(T)$ , where

$$\Gamma(T) := \left\{ \begin{pmatrix} x \\ Tx \end{pmatrix}, x \in \mathbb{C}^n \right\}.$$

Keeping this in mind, the following definitions are quite familiar.

**Definition 2.1.** For linear relations  $S, T \subseteq \mathbb{C}^{2n}$  we define

$$\begin{aligned} \text{dom } S &= \left\{ x : \begin{pmatrix} x \\ y \end{pmatrix} \in S \right\}, & \text{the domain of } S; \\ \text{mul } S &= \left\{ y : \begin{pmatrix} 0 \\ y \end{pmatrix} \in S \right\}, & \text{the multivalued part of } S; \\ S^{-1} &= \left\{ \begin{pmatrix} y \\ x \end{pmatrix} : \begin{pmatrix} x \\ y \end{pmatrix} \in S \right\}, & \text{the inverse of } S; \\ S + T &= \left\{ \begin{pmatrix} x \\ y+z \end{pmatrix} : \begin{pmatrix} x \\ y \end{pmatrix} \in S, \begin{pmatrix} x \\ z \end{pmatrix} \in T \right\}, & \text{the sum of } S, \text{ and } T; \end{aligned}$$

and the *product* of  $S$  and  $T$

$$ST = \left\{ \begin{pmatrix} x \\ z \end{pmatrix} : \text{there exists a } y \in \mathbb{C}^n \text{ with } \begin{pmatrix} y \\ z \end{pmatrix} \in S, \begin{pmatrix} x \\ y \end{pmatrix} \in T \right\}.$$

If  $\text{dom } S = \mathbb{C}^n$ , we say that  $S$  has full domain. In all the cases  $x, y, z$  are understood to be from  $\mathbb{C}^n$ .

Note that a linear relation is always invertible in the above sense. We now give a more general definition of the *H-adjoint* of a linear relation  $T$ . This coincides with the linear relation in (5) when  $T$  is a matrix.

**Definition 2.2.** Let  $[\cdot, \cdot]$  denote the indefinite inner product induced by  $H$ , and let  $T$  be a linear relation in  $\mathbb{C}^n$ . Then the linear relation

$$T^{[*]} = \left\{ \begin{pmatrix} y \\ z \end{pmatrix} \in \mathbb{C}^{2n} : [y, w] = [z, x] \text{ for all } \begin{pmatrix} x \\ w \end{pmatrix} \in T \right\}$$

is called the *H-adjoint* of  $T$ .

The next proposition which is a summary of Proposition 2.2 and Lemma 2.4 of [16] contains some basic properties of the *H-adjoint*.

**Proposition 2.3.** Let  $S, T \subseteq \mathbb{C}^{2n}$  be linear relations. Then

$$(i) \ S \subseteq T \text{ implies } T^{[*]} \subseteq S^{[*]};$$

- (ii)  $S^{[*]} + T^{[*]} \subseteq (S + T)^{[*]}$ ;
- (iii)  $T^{[*]}S^{[*]} \subseteq (ST)^{[*]}$ ;
- (iv)  $\text{mul } T^{[*]} = (\text{dom } T)^{[\perp]}$ ; if  $T$  is a matrix, then  $\text{mul } T^{[*]} = \ker H$ ;
- (v)  $(T^{[*]})^{[*]} = T + (\ker H \times \ker H)$ .

If  $T \in \mathbb{C}^{n \times n}$  is a matrix, then

$$T^{[*]} = \left\{ \begin{pmatrix} y \\ z \end{pmatrix} \in \mathbb{C}^{2n} : T^*Hy = Hz \right\}. \quad (6)$$

In particular,  $T^{[*]}$  is a matrix if and only if  $H$  is invertible.

Note that we can always find a basis of  $\mathbb{C}^n$  such that the matrices  $H$  and  $T$  have the forms

$$H = \begin{bmatrix} H_1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad T = \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix} \in \mathbb{C}^{n \times n}, \quad (7)$$

where  $H_1, T_1 \in \mathbb{C}^{m \times m}$ ,  $m \leq n$ , and  $H_1$  is invertible. Using this decomposition,  $T$  can be written as

$$T = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ T_1x_1 + T_2x_2 \\ T_3x_1 + T_4x_2 \end{pmatrix} : x_1 \in \mathbb{C}^m, x_2 \in \mathbb{C}^{n-m} \right\} = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ T_1x_1 + T_2x_2 \\ T_3x_1 + T_4x_2 \end{pmatrix} \right\}.$$

We will use the short second notion when it is clear from the context how the matrices  $T$  and  $H$  are decomposed.

**Proposition 2.4** ([16]). Let  $T \in \mathbb{C}^{n \times n}$  be a matrix. Then

$$T^{[*]} = H^{-1}T^*H,$$

where  $H^{-1}$  denotes the inverse of  $H$  in the sense of linear relations. Furthermore, if  $H$  and  $T$  have the forms as in (7), then

$$T^{[*]} = \left\{ \begin{pmatrix} y_1 \\ y_2 \\ T_1^{[*]H_1}y_1 \\ z_2 \end{pmatrix} : T_2^*H_1y_1 = 0 \right\}, \quad (8)$$

where  $T_1^{[*]H_1}$  denotes the adjoint with respect to the invertible matrix  $H_1$ , i.e.,  $T_1^{[*]H_1} = H_1^{-1}T_1^*H_1$ . In particular,  $\text{dom } T^{[*]} = \mathbb{C}^n$  if and only if  $T_2 = 0$ .

We will suppress the subscript  $H_1$ , writing  $T_1^{[*]}$  instead of  $T_1^{[*]H_1}$  when it is clear from the context that  $H_1$  induces the indefinite inner product.

## 2.2. $H$ -Symmetric linear relations

As usual, a linear relation  $T \subseteq \mathbb{C}^{2n}$  is called  $H$ -symmetric if  $T \subseteq T^{[*]}$ , see, e.g., [19]. The notion of  $H$ -symmetric linear relations will be needed when we introduce the class of  $H$ -hyponormal matrices in Section 3. We therefore collect some of the basic properties of  $H$ -symmetric matrices and linear relations from [16, Propositions 3.3 and 3.4, Corollary 3.5].

**Proposition 2.5.** Let  $T \in \mathbb{C}^{n \times n}$  be a matrix. Then the following statements are equivalent:

- (i)  $T$  is  $H$ -symmetric, i.e.,  $T \subseteq T^{[*]}$ ;
- (ii)  $T^*H = HT$ ;

- (iii)  $T^{[*]} = (T^{[*]})^{[*]}$ ;
- (iv)  $T^{[*]} = T + (\ker H \times \ker H)$ .

If one of the conditions is satisfied, then  $\ker H$  is  $T$ -invariant. In particular, if  $H$  and  $T$  have the forms as in (7) then  $T$  is  $H$ -symmetric if and only if  $T_1$  is  $H_1$ -selfadjoint and  $T_2 = 0$ .

Recall from Proposition 2.3 that in the case of a singular  $H$  the relation  $T^{[*]}$  is never a matrix. Hence, a matrix with  $T = T^{[*]}$  does not exist. Therefore, in view of Theorem 2.5(iii),  $H$ -symmetry can be considered as a generalization of the notion of  $H$ -selfadjointness for the case of a singular  $H$ . The following proposition will be an important tool in Section 3.

**Proposition 2.6.** Let  $T \subset \mathbb{C}^{2n}$  be a linear relation:

- (1)  $TT^{[*]}$  and  $T^{[*]}T$  are  $H$ -symmetric, i.e.,  
 $TT^{[*]} \subseteq (TT^{[*]})^{[*]} \quad \text{and} \quad T^{[*]}T \subseteq (T^{[*]}T)^{[*]}.$
- (2) If  $T \in \mathbb{C}^{n \times n}$  is a matrix, then the following assertions are equivalent:
  - (i) The domain of the linear relation  $T^{[*]}T$  is  $\mathbb{C}^n$ ,
  - (ii)  $T^{[*]}T = (T^{[*]}T)^{[*]}.$

In particular, if  $T$  and  $H$  are in the form (7), then (i) and (ii) are equivalent to

- (iii)  $T_2^* H_1 T_1 = 0$  and  $T_2^* H_1 T_2 = 0$ .

**Proof.** (1) Proposition 2.3(v) implies  $T \subseteq (T^{[*]})^{[*]}$ . Hence, by Proposition 2.3(iii), we have that

$$TT^{[*]} \subseteq (T^{[*]})^{[*]}T^{[*]} \subseteq (TT^{[*]})^{[*]}.$$

Analogously, we obtain

$$T^{[*]}T \subseteq T^{[*]}(T^{[*]})^{[*]} \subseteq (T^{[*]}T)^{[*]}.$$

(2) Assume now that  $T \in \mathbb{C}^{n \times n}$  is a matrix. Without loss of generality, we may assume that  $T$  and  $H$  are in the form (7). Then Proposition 2.4 implies

$$T^{[*]}T = \left\{ \begin{pmatrix} y_1 \\ y_2 \\ T_1^{[*]}T_1y_1 + T_1^{[*]}T_2y_2 \\ z_2 \end{pmatrix} : T_2^*H_1T_1y_1 + T_2^*H_1T_2y_2 = 0 \right\}. \quad (9)$$

Hence, (iii) and (i) are equivalent. If (iii) holds, we have

$$0 = (T_2^*H_1T_1)^* = T_1^*H_1T_2 \quad \text{and} \quad 0 = H_1^{-1}T_1^*H_1T_2 = T_1^{[*]}T_2.$$

By a simple calculation together with (9) we obtain

$$T^{[*]}T = \left\{ \begin{pmatrix} y_1 \\ y_2 \\ T_1^{[*]}T_1y_1 \\ z_2 \end{pmatrix} \right\} = (T^{[*]}T)^{[*]}$$

and (ii) follows.

For the remainder of the proof we assume that (ii) holds. If there exists  $y_2$  with  $T_2^*H_1T_2y_2 \neq 0$ , then, by (9),  $\begin{pmatrix} 0 \\ y_2 \end{pmatrix} \notin \text{dom } T^{[*]}T$ . But

$$\begin{pmatrix} 0 \\ y_2 \\ 0 \\ 0 \end{pmatrix} \in (T^{[*]}T)^{[*]}$$

a contradiction to (ii). Therefore, (ii) implies  $T_2^* H_1 T_2 y_2 = 0$  and we have by (9) that

$$T^{[*]}T = \left\{ \begin{pmatrix} y_1 \\ y_2 \\ T_1^{[*]}T_1 y_1 + T_1^{[*]}T_2 y_2 \\ z_2 \end{pmatrix} : T_2^* H_1 T_1 y_1 = 0 \right\}.$$

Now let  $w_1 \in (\ker T_2^* H_1 T_1)^{\perp_{H_1}}$ , that is  $[w_1, y_1] = 0$  for all  $y_1$  satisfying  $T_2^* H_1 T_1 y_1 = 0$ . Then

$$\begin{pmatrix} 0 \\ 0 \\ w_1 \\ 0 \end{pmatrix} \in (T^{[*]}T)^{[*]} = T^{[*]}T,$$

hence  $w_1 = 0$  and  $T_2^* H_1 T_1 y_1 = 0$  follows. Thus (ii) implies (iii).  $\square$

**Remark 2.7.** We mention that  $TT^{[*]}$  has full domain if and only if  $T^{[*]}$  has full domain, which is equivalent to the fact that  $T_2 = 0$ , cf. Proposition 2.4. However, a similar statement as the equivalence of (i) and (ii) in Proposition 2.6, part 2, does not hold for  $TT^{[*]}$ . As an example consider the matrix  $T = 0$ . Then  $TT^{[*]}$  is the zero matrix but  $\text{mul}(TT^{[*]})^{[*]} = \ker H$ , see Proposition 2.3. Hence  $TT^{[*]}$  has full domain but  $TT^{[*]} \neq (TT^{[*]})^{[*]}$ .

### 2.3. $H$ -Normal matrices

In the case of a singular  $H$ , where the matrices  $H$  and  $T$  are given in the forms as in (7), it is easily checked that

$$TT^{[*]} = \left\{ \begin{pmatrix} y_1 \\ y_2 \\ T_1 T_1^{[*]} y_1 + T_2 y_2 \\ T_3 T_1^{[*]} y_1 + T_4 z_2 \end{pmatrix} : T_2^* H_1 y_1 = 0 \right\}. \quad (10)$$

Comparing (9) and (10) one can easily see that in the case of an  $H$ -symmetric  $T$  we obtain only that  $TT^{[*]} \subseteq T^{[*]}T$  and the inclusion  $T^{[*]}T \subseteq TT^{[*]}$  is only satisfied in the case that  $T_4$  is invertible. Therefore,  $H$ -normal matrices were defined in [16] by the inclusion

$$TT^{[*]} \subseteq T^{[*]}T \quad (11)$$

rather than by the identity  $TT^{[*]} = T^{[*]}T$ , because otherwise there would exist  $T$ -symmetric matrices that are not  $H$ -normal. It was then shown in [16, Proposition 4.2] that for  $H$ -normal matrices  $\ker H$  is  $T$ -invariant and that, if  $T$  and  $H$  are given in the forms (7),  $T$  is  $H$ -normal if and only if  $T_1$  is  $H_1$ -normal and  $T_2 = 0$ .

### 3. $H$ -Hyponormal matrices

If the Hermitian matrix  $H \in \mathbb{C}^{n \times n}$  is invertible, then an  $H$ -hyponormal matrix  $T$  by definition satisfies

$$H(T^{[*]}T - TT^{[*]}) \geq 0,$$

i.e.,  $T^{[*]}T - TT^{[*]}$  is  $H$ -nonnegative. Such matrices were discussed, e.g., in [13,14]. A generalization of this definition to the case of singular  $H$  requires the concept of  $H$ -nonnegativity for linear relations.

**Definition 3.1.** A linear relation  $S \subseteq \mathbb{C}^{2n}$  is called  $H$ -nonnegative if  $S$  is  $H$ -symmetric (i.e.  $S \subseteq S^{[*]}$ ) and if

$$[y, x] \geq 0 \quad \text{for all } \begin{pmatrix} x \\ y \end{pmatrix} \in S. \quad (12)$$

Analogously the notions of  $H$ -nonpositivity,  $H$ -positivity and  $H$ -negativity of a linear relation are defined. The following lemma yields a base for the definition of  $H$ -hyponormal matrices.

**Lemma 3.2.** *Let  $T \subseteq \mathbb{C}^{2n}$  be a linear relation. Then the relation  $T^{[*]}T - TT^{[*]}$  is  $H$ -symmetric.*

**Proof.** Propositions 2.6 and 2.3(ii) imply

$$T^{[*]}T - TT^{[*]} \subseteq (T^{[*]}T)^{[*]} - (TT^{[*]})^{[*]} \subseteq (T^{[*]}T - TT^{[*]})^{[*]}. \quad \square$$

In the following, we will give conditions which ensure the  $H$ -nonnegativity of  $T^{[*]}T - TT^{[*]}$ . In the case that  $T$  is a matrix, the following characterization holds.

**Proposition 3.3.** *Let  $T \in \mathbb{C}^{n \times n}$  be a matrix and let  $T$  and  $H$  be in the forms (7). Then  $T^{[*]}T - TT^{[*]}$  is  $H$ -nonnegative if and only if*

$$y_1^* H_1 (T_1^{[*]} T_1 - T_1 T_1^{[*]}) y_1 \geq y_2^* T_2^* H_1 T_2 y_2$$

for all  $y_1, y_2$  satisfying  $T_2^* H_1 y_1 = 0$  and  $T_2^* H_1 (T_1 y_1 + T_2 y_2) = 0$ .

**Proof.** The linear relation  $T^{[*]}T - TT^{[*]}$  is  $H$ -symmetric by Lemma 3.2. Then we obtain from (9) and (10) that  $T^{[*]}T - TT^{[*]}$  equals

$$\left\{ \begin{pmatrix} y_1 \\ y_2 \\ (T_1^{[*]} T_1 - T_1 T_1^{[*]}) y_1 + T_1^{[*]} T_2 y_2 - T_2 y_2 \\ w_2 - T_3 T_1^{[*]} y_1 - T_4 z_2 \end{pmatrix} : \begin{matrix} T_2^* H_1 y_1 = 0; \\ T_2^* H_1 (T_1 y_1 + T_2 y_2) = 0 \end{matrix} \right\}.$$

Thus,  $T^{[*]}T - TT^{[*]}$  is  $H$ -nonnegative if and only if

$$y_1^* H_1 (T_1^{[*]} T_1 - T_1 T_1^{[*]}) y_1 + y_1^* H_1 T_1^{[*]} T_2 y_2 - y_1^* H_1 T_2 y_2 \geq 0 \quad (13)$$

for all  $y_1, y_2$  that satisfy  $T_2^* H_1 y_1 = 0$  and  $T_2^* H_1 (T_1 y_1 + T_2 y_2) = 0$ . The restrictions on  $y_1$  and  $y_2$  imply

$$y_1^* H_1 T_2 y_2 = 0 \quad \text{and} \quad y_1^* H_1 T_1^{[*]} T_2 y_2 = y_1^* T_1^* H_1 T_2 y_2 = -y_2^* T_2^* H_1 T_2 y_2.$$

Thus, (13) reduces to  $y_1^* H_1 (T_1^{[*]} T_1 - T_1 T_1^{[*]}) y_1 \geq y_2^* T_2^* H_1 T_2 y_2$ .  $\square$

At this point, one could define  $H$ -hyponormal matrices as matrices  $T$  for which the linear relation  $T^{[*]}T - TT^{[*]}$  is  $H$ -nonnegative. However, this would not be satisfactory, because the class of matrices obtained in this way is too general. In particular, an important property of  $H$ -hyponormal matrices would be lost. It is well known that if  $H$  is negative definite then  $H$ -hyponormality implies  $H$ -normality, see, e.g., [6]. However, if we relax the condition of  $H$  being negative definite to  $H$  being negative semidefinite, then  $H$ -nonnegativity of  $T^{[*]}T - TT^{[*]}$  is no longer sufficient for  $H$ -normality as the following example shows.

**Example 3.4.** Let  $r, m, n$  be such that  $r < m, r + m < n$  and let

$$H = \begin{bmatrix} -I_m & 0 \\ 0 & 0 \end{bmatrix}, \quad T = \left[ \begin{array}{c|c} T_1 & T_2 \\ \hline T_3 & T_4 \end{array} \right] = \left[ \begin{array}{cc|cc} T_{11} & T_{12} & I_r & 0 \\ T_{13} & T_{14} & 0 & 0 \\ \hline T_{31} & T_{32} & T_{41} & T_{42} \\ T_{33} & T_{34} & T_{43} & T_{44} \end{array} \right],$$

where  $H \in \mathbb{C}^{n \times n}$ ,  $T_{11} \in \mathbb{C}^{r \times r}$ ,  $T_{41} \in \mathbb{C}^{r \times r}$ ,  $T_{44} \in \mathbb{C}^{(n-r-m) \times (n-r-m)}$ , and where  $T_{14} \in \mathbb{C}^{(m-r) \times (m-r)}$  is normal with respect to the standard Euclidean product, i.e.,  $T_{14}^* T_{14} = T_{14} T_{14}^*$ . Moreover, let

$$y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} y_{11} \\ y_{12} \\ y_{21} \\ y_{22} \end{pmatrix}$$



be partitioned conformably with  $T$ . Then by (9) and (10),  $y$  is in the domain of  $T^{[*]}T - TT^{[*]}$  if and only if

$$T_2^* H_1 y_1 = 0 \quad \text{and} \quad T_2^* H_1 (T_1 y_1 + T_2 y_2) = 0. \quad (14)$$

The first identity implies that

$$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -I_r & 0 \\ 0 & -I_{m-r} \end{bmatrix} \begin{pmatrix} y_{11} \\ y_{12} \end{pmatrix} = \begin{pmatrix} -y_{11} \\ 0 \end{pmatrix} = 0.$$

Thus, if  $y$  is in the domain of  $T^{[*]}T - TT^{[*]}$ , then  $y_{11} = 0$ . The second identity in (14) implies

$$T_2^* H_1 T_1 y_1 = -T_2^* H_1 T_2 y_2, \quad (15)$$

that is

$$\begin{pmatrix} -T_{11}y_{11} - T_{12}y_{12} \\ 0 \end{pmatrix} = - \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -I_r & 0 \\ 0 & -I_{m-r} \end{bmatrix} \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} y_{21} \\ y_{22} \end{pmatrix},$$

therefore

$$\begin{pmatrix} -T_{11}y_{11} - T_{12}y_{12} \\ 0 \end{pmatrix} = \begin{pmatrix} y_{21} \\ 0 \end{pmatrix}.$$

Hence  $y$  is in the domain of  $T^{[*]}T - TT^{[*]}$  if and only if it has the form:

$$y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ y_{12} \\ -T_{12}y_{12} \\ y_{22} \end{pmatrix}. \quad (16)$$

Due to the normality of  $T_{14}$  and because of (15), we obtain that

$$\begin{aligned} y_1^* H_1 (T_1^{[*]} T_1 - T_1 T_1^{[*]}) y_1 &= -y_{12}^* (T_{12}^* T_{12} + T_{14}^* T_{14} - T_{13} T_{13}^* - T_{14} T_{14}^*) y_{12} \\ &= -y_{12}^* (T_{12}^* T_{12} - T_{13} T_{13}^*) y_{12} \geq -y_{12}^* T_{12}^* T_{12} y_{12} = y_2^* T_2^* H_1 T_2 y_2. \end{aligned}$$

Thus by Proposition 3.3,  $T^{[*]}T - TT^{[*]}$  is  $H$ -nonnegative. However, if  $r > 0$  then  $T$  is not  $H$ -normal, because the kernel of  $H$  is not  $T$ -invariant, see [16, Proposition 4.2].

Moreover, in the above Example 3.4 we have  $T_2^* H_1 T_2 \neq 0$  and by [16, Relation (4.6) in the proof of Proposition 4.6] the matrix  $T$  is not Moore–Penrose  $H$ -normal. Recall that a matrix is called Moore–Penrose  $H$ -normal if

$$HTH^\dagger T^* H = T^* HT,$$

where  $H^\dagger$  is the Moore–Penrose generalized inverse of  $H$ , see also Section 5 below.

Example 3.4 shows that the reason for the failure of the desired property is the domain of  $T^{[*]}T - TT^{[*]}$  which is too small, in general. One way to circumvent this problem is to require that the linear relation  $TT^{[*]}$  has full domain. From (10) it follows that this is equivalent to  $T_2 = 0$  (if  $H$  and  $T$  are assumed to be in the form (7)) and thus, we then also have that  $T^{[*]}T$  and hence  $T^{[*]}T - TT^{[*]}$  have full domain. However, this condition is rather restrictive as  $T_2 = 0$  implies that the kernel of  $H$  is  $T$ -invariant. Consequently, the set of matrices obtained in this way does not contain all Moore–Penrose  $H$ -normal matrices (we refer to [12, Example 6.1] for a Moore–Penrose  $H$ -normal matrix such that  $\ker H$  is not invariant). Fortunately, it turns out that it is enough to require that  $T^{[*]}T$  has full domain in order to guarantee that the domain of  $T^{[*]}T - TT^{[*]}$  is sufficiently large so that  $H$ -nonnegativity of  $T^{[*]}T - TT^{[*]}$  implies  $H$ -normality in the case of negative semidefinite  $H$ . This motivates the following definition of  $H$ -hyponormality.

**Definition 3.5.** A linear relation  $T \subseteq \mathbb{C}^{2n}$  is called *H-hyponormal* if  $T^{[*]}T$  has full domain and if  $T^{[*]}T - TT^{[*]}$  is *H*-nonnegative.

From Propositions 3.3 and 2.6, we immediately obtain the following characterization of *H*-hyponormal matrices.

**Proposition 3.6.** Let  $T \in \mathbb{C}^{n \times n}$  be a matrix and let  $T$  and  $H$  be in the forms (7). Then  $T$  is *H*-hyponormal if and only if  $T^{[*]}T$  has full domain and

$$y_1^* H_1 (T_1^{[*]} T_1 - T_1 T_1^{[*]}) y_1 \geq 0$$

for all  $y_1$  satisfying  $T_2^* H_1 y_1 = 0$ .

As a corollary, we obtain the desired property that *H*-hyponormality is equivalent to *H*-normality for negative semidefinite matrices  $H$ .

**Corollary 3.7.** Let  $H \in \mathbb{C}^{n \times n}$  be negative semidefinite and let  $T \in \mathbb{C}^{n \times n}$  be an *H*-hyponormal matrix. Then  $T$  is *H*-normal.

**Proof.** Without loss of generality let  $T$  and  $H$  be in the forms (7), where  $H_1 = -I_m$ . By Proposition 2.6, we have  $-T_2^* T_2 = T_2^* H_1 T_2 = 0$  which implies that  $T_2 = 0$ . By Proposition 3.6, we then obtain that

$$-y_1^* (T_1^* T_1 - T_1 T_1^*) y_1 \geq 0$$

for all  $y_1 \in \mathbb{C}^m$ , that is,  $T_1^* T_1 - T_1 T_1^*$  is negative semidefinite. Thus, all eigenvalues of  $T_1^* T_1 - T_1 T_1^*$  are smaller or equal to zero, and as  $\text{tr}(T_1^* T_1 - T_1 T_1^*) = 0$  it follows that  $T_1^* T_1 - T_1 T_1^* = 0$  and hence  $T_1$  is normal. From Proposition [16, Proposition 4.2] we then get that  $T$  is *H*-normal.  $\square$

In [16, Proposition 4.2], it was shown that the kernel of  $H$  is always  $T$ -invariant if  $T$  is *H*-normal, and it was shown in [12] that the kernel of  $H$  is always contained in a  $T$ -invariant *H*-neutral subspace if  $T$  is Moore–Penrose *H*-normal. Unfortunately, this is no longer true for *H*-hyponormal matrices as the following example shows.

**Example 3.8.** Let

$$H = \left[ \begin{array}{ccc|c} H_1 & 0 \\ \hline 0 & 0 \end{array} \right] = \left[ \begin{array}{ccc|c} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right], \quad T = \left[ \begin{array}{cc|c} T_1 & T_2 \\ \hline 0 & 0 \end{array} \right] = \left[ \begin{array}{ccc|c} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right].$$

Then one computes that  $T_2^* H_1 T_2 = 0$  and  $T_2^* H_1 T_1 = 0$  and, by Proposition 2.6,  $T^{[*]}T$  has full domain. Moreover,

$$T_1^{[*]} = \left[ \begin{array}{ccc} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right] \quad \text{and} \quad H_1 (T_1^{[*]} T_1 - T_1 T_1^{[*]}) = \left[ \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right]$$

hence  $y^* H_1 (T_1^{[*]} T_1 - T_1 T_1^{[*]}) y \geq 0$  for all  $y \in \mathbb{C}^3$  and, by Proposition 3.6, we obtain that  $T$  is *H*-hyponormal. However, note that  $\ker H = \text{span}(e_4)$  and that  $\mathcal{U} := \text{span}(e_4, e_1 + e_3, e_2)$  is the smallest  $T$ -invariant subspace containing  $\ker H$ . (Here,  $e_i$  denotes the  $i$ -th standard basis vector.) Obviously,  $\mathcal{U}$  is not *H*-neutral, as  $e_2^* H e_2 = 1$ .

#### 4. Strongly *H*-hyponormal matrices

We have seen in the previous section that *H*-hyponormal matrices are too general. We will therefore define a new class of matrices which is properly contained in the set of *H*-hyponormal matrices and small enough to ensure that the kernel of  $H$  is contained in an invariant *H*-neutral subspace.

**Definition 4.1.** Let  $T \subseteq \mathbb{C}^{2n}$  be a linear relation:

- (1)  $T$  is called *strongly  $H$ -hyponormal of degree  $k \in \mathbb{N}$*  if  $T$  is  $H$ -hyponormal and if  $(T^{[*]})^i T^i$  has full domain for all  $i = 1, 2, \dots, k$ .
- (2)  $T$  is called *strongly  $H$ -hyponormal* if  $T$  is strongly  $H$ -hyponormal of degree  $k$  for all  $k \in \mathbb{N}$ .

We start with two examples which show that the class of strongly  $H$ -hyponormal matrices neither coincides with the class of  $H$ -normal matrices nor with the class of  $H$ -hyponormal matrices.

**Example 4.2.** Let

$$H = \left[ \begin{array}{c|c} H_1 & 0 \\ \hline 0 & 0 \end{array} \right] = \left[ \begin{array}{cc|c} -1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 0 \end{array} \right], \quad T = \left[ \begin{array}{c|c} T_1 & T_2 \\ \hline 0 & 0 \end{array} \right] = \left[ \begin{array}{cc|c} 1 & 0 & 1 \\ 1 & 0 & 1 \\ \hline 0 & 0 & 0 \end{array} \right].$$

Then  $T_2^* H_1 T_2 = 0$  and  $T_2^* H_1 T_1 = 0$ , that is,  $T^{[*]} T$  has full domain. Moreover,

$$T_1^{[*]} = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad H_1 (T_1^{[*]} T_1 - T_1 T_1^{[*]}) = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

Hence  $H_1 (T_1^{[*]} T_1 - T_1 T_1^{[*]})$  is positive semidefinite, i.e.,

$$y_1^* H_1 (T_1^{[*]} T_1 - T_1 T_1^{[*]}) y_1 \geq 0 = y_2^* T_2^* H_1 T_2 y_2$$

for all  $y_1, y_2$ . Thus,  $T$  is  $H$ -hyponormal by Proposition 3.3. Moreover,  $T$  is also strongly  $H$ -hyponormal, because  $T$  is idempotent and, by Proposition 2.4,

$$T^{[*]} = \left\{ \begin{pmatrix} \alpha \\ \alpha \\ \beta \\ 0 \\ 0 \\ \gamma \end{pmatrix} : \alpha, \beta, \gamma \in \mathbb{C} \right\}.$$

Hence,  $(T^{[*]})^k = T^{[*]}$  and  $T^k = T$ ,  $k \in \mathbb{N}$ , so that  $(T^{[*]})^k T^k = T^{[*]} T$  has full domain for all  $k \in \mathbb{N}$ . In particular,  $T$  is an example for a matrix that is strongly  $H$ -hyponormal, but not  $H$ -normal, because  $T_2 \neq 0$ , i.e.,  $\ker H$  is not  $T$ -invariant, cf. [16, Proposition 4.2].

**Example 4.3.** The matrix given in Example 3.8 is not strongly  $H$ -hyponormal. Indeed, by Proposition 2.4,

$$T^{[*]} = \{(y_1, y_2, y_1, y_4, -y_2, 0, 0, z_4)^T : y_1, y_2, y_4, z_4 \in \mathbb{C}\}$$

and one easily checks that

$$\text{dom}(T^{[*]})^2 = \left\{ \begin{pmatrix} y_1 \\ 0 \\ y_1 \\ y_4 \end{pmatrix} : y_1, y_4 \in \mathbb{C} \right\}$$

and

$$\text{dom}(T^{[*]})^2 T^2 = \left\{ \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ 0 \end{pmatrix} : y_1, y_2, y_3 \in \mathbb{C} \right\}.$$

That is,  $T$  is  $H$ -hyponormal (see Example 3.8) but not strongly  $H$ -hyponormal.

In the following proposition, we characterize the property that  $(T^{[*]})^i T^i$  has full domain in terms of  $T_1$ ,  $T_2$ , and  $H_1$  when  $T$  and  $H$  are in the forms (7). We will use this characterization in the proof of our main result of this section, Theorem 4.5 below.

**Proposition 4.4.** *Let  $T \subseteq \mathbb{C}^{2n}$  be a linear relation. Then  $(T^{[*]})^k T^k$  and  $T^k (T^{[*]})^k$  are  $H$ -symmetric for all  $k \in \mathbb{N}$ . In particular, if  $T \in \mathbb{C}^{n \times n}$  is a matrix and if  $T$  and  $H$  are in the forms (7), then the following assertions are equivalent:*

- (1)  $(T^{[*]})^i T^i$  has full domain for  $1 \leq i \leq k$ ;
- (2)  $T_2^* H_1 (T_1^{[*]})^{i-1} T_1^{i-1} T_1 = 0$  and  $T_2^* H_1 (T_1^{[*]})^{i-1} T_1^{i-1} T_2 = 0$  for  $1 \leq i \leq k$ .

**Proof.** We first show that  $T^i (T^{[*]})^i$  is  $H$ -symmetric, that is

$$T^i (T^{[*]})^i \subseteq (T^i (T^{[*]})^i)^{[*]}.$$

By Proposition 2.3(iii) we find that  $(T^{[*]})^i \subseteq (T^i)^{[*]}$  and therefore

$$T^i (T^{[*]})^i \subseteq T^i (T^i)^{[*]}. \quad (17)$$

From Propositions 2.3(v) and 2.3(iii), (i) it follows that

$$T^i \subseteq ((T^i)^{[*]})^{[*]} \quad \text{and} \quad ((T^i)^{[*]})^{[*]} \subseteq ((T^{[*]})^i)^{[*]}.$$

Hence  $T^i \subseteq ((T^{[*]})^i)^{[*]}$ , and thus

$$T^i (T^i)^{[*]} \subseteq ((T^{[*]})^i)^{[*]} (T^i)^{[*]}. \quad (18)$$

Putting together (17), (18), and using Proposition 2.3(iii), we obtain that

$$T^i (T^{[*]})^i \subseteq (T^i (T^{[*]})^i)^{[*]},$$

thus  $T^i (T^{[*]})^i$  is  $H$ -symmetric. A similar argumentation shows that  $(T^{[*]})^i T^i$  is also  $H$ -symmetric.

We next show by induction on  $k$  that

$$T^k = \begin{bmatrix} T_1^k + B_k & T_1^{k-1} T_2 + C_k \\ * & * \end{bmatrix}, \quad (19)$$

where

$$B_k = \sum_{i=2}^k T_1^{k-i} T_2 D_i^{(k)} \quad \text{and} \quad C_k = \sum_{i=2}^k T_1^{k-i} T_2 \tilde{D}_i^{(k)}$$

for some matrices  $D_i^{(k)} \in \mathbb{C}^{(n-m) \times m}$  and  $\tilde{D}_i^{(k)} \in \mathbb{C}^{(n-m) \times (n-m)}$ . For the case  $k = 1$  there is nothing to show as  $B_1 = 0$  and  $C_1 = 0$  by the definition of the empty sum. If  $k \geq 2$  then

$$\begin{aligned} T^{k+1} &= \begin{bmatrix} T_1^k + B_k & T_1^{k-1} T_2 + C_k \\ * & * \end{bmatrix} \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix} \\ &= \begin{bmatrix} T_1^{k+1} + B_k T_1 + T_1^{k-1} T_2 T_3 + C_k T_3 & T_1^k T_2 + B_k T_2 + T_1^{k-1} T_2 T_4 + C_k T_4 \\ * & * \end{bmatrix}. \end{aligned}$$

By the induction hypothesis, we find that

$$B_k T_1 + T_1^{k-1} T_2 T_3 + C_k T_3 = \sum_{i=2}^k T_1^{k-i} T_2 D_i^{(k)} T_1 + T_1^{k-1} T_2 T_3 + \sum_{i=2}^k T_1^{k-i} T_2 \tilde{D}_i^{(k)} T_3$$

and

$$B_k T_2 + T_1^{k-1} T_2 T_4 + C_k T_4 = \sum_{i=2}^k T_1^{k-i} T_2 D_i^{(k)} T_2 + T_1^{k-1} T_2 T_4 + \sum_{i=2}^k T_1^{k-i} T_2 \tilde{D}_i^{(k)} T_4.$$

Thus, by setting

$$D_2^{(k+1)} := T_3, \quad D_i^{(k+1)} := D_{i-1}^{(k)} T_1 + \tilde{D}_{i-1}^{(k)} T_3 \quad \text{for } i = 3, \dots, k+1$$

and

$$\tilde{D}_2^{(k+1)} := T_4, \quad \tilde{D}_i^{(k+1)} := D_{i-1}^{(k)} T_2 + \tilde{D}_{i-1}^{(k)} T_4 \quad \text{for } i = 3, \dots, k+1,$$

we obtain that  $D_i^{(k+1)} \in \mathbb{C}^{(n-m) \times n}$  and  $\tilde{D}_i^{(k+1)} \in \mathbb{C}^{(n-m) \times (n-m)}$ , and therefore by setting

$$B_{k+1} = \sum_{i=2}^{k+1} T_1^{k+1-i} T_2 D_i^{(k+1)} \quad \text{and} \quad C_{k+1} = \sum_{i=2}^{k+1} T_1^{k+1-i} T_2 \tilde{D}_i^{(k+1)}, \quad (20)$$

we have

$$T^{k+1} = \begin{bmatrix} T_1^{k+1} + B_{k+1} & T_1^k T_2 + C_{k+1} \\ * & * \end{bmatrix}$$

as desired.

We now prove the equivalence of (1) and (2) by induction on  $k$ . The case  $k = 1$  is covered by Proposition 2.6. If  $k \geq 2$ , note that for  $j = 1, \dots, k+1$  we obtain using Proposition 2.4 that

$$(T^{[*]})^j T^{k+1} = \left\{ \begin{pmatrix} y_1 \\ y_2 \\ (T_1^{[*]})^j W_{k+1} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \\ z_2 \end{pmatrix} : \begin{array}{l} T_2^* H_1 (T_1^{[*]})^s W_{k+1} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = 0; \\ s = 0, \dots, j-1 \end{array} \right\}, \quad (21)$$

where  $W_{k+1} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$  is the first component of the vector  $T^{k+1} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ , i.e.,

$$W_{k+1} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = (T_1^{k+1} + B_{k+1}) y_1 + (T_1^k T_2 + C_{k+1}) y_2.$$

Now assume that either (1) or (2) holds for  $1 \leq i \leq k+1$ . Using the equivalence of conditions postulated in the induction hypothesis, we have in either case that  $(T^{[*]})^i T^i$  has full domain for  $1 \leq i \leq k$  and that

$$T_2^* H_1 (T_1^{[*]})^{i-1} T_1^{i-1} T_1 = 0, \quad T_2^* H_1 (T_1^{[*]})^{i-1} T_1^{i-1} T_2 = 0 \quad \text{for } 1 \leq i \leq k. \quad (22)$$

Thus it remains to show that the following assertions are equivalent:

- (a)  $(T^{[*]})^{k+1} T^{k+1}$  has full domain;
- (b)  $T_2^* H_1 (T_1^{[*]})^k T_1^k T_1 = 0$  and  $T_2^* H_1 (T_1^{[*]})^k T_1^k T_2 = 0$ .

It follows from (21) that the relation  $(T^{[*]})^{k+1} T^{k+1}$  has full domain if and only if  $(T^{[*]})^j T^{k+1}$  has full domain for all  $j = 1, \dots, k+1$ . We will first show that (22) implies that  $(T^{[*]})^j T^{k+1}$  has full domain for  $j = 1, \dots, k$  that is  $T_2^* H_1 (T_1^{[*]})^{j-1} W_{k+1} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = 0$  for all  $y_1, y_2$  and all  $j = 1, \dots, k$ . This means that we have to show the following:

- (i)  $T_2^* H_1 (T_1^{[*]})^{j-1} T_1^{j-1} y_1 = 0$ ;
- (ii)  $T_2^* H_1 (T_1^{[*]})^{j-1} B_{k+1} y_1 = 0$ ;
- (iii)  $T_2^* H_1 (T_1^{[*]})^{j-1} T_1^k T_2 y_2 = 0$ ;
- (iv)  $T_2^* H_1 (T_1^{[*]})^{j-1} C_{k+1} y_2 = 0$ .

We easily obtain (i) and (iii) as

$$T_2^* H_1 (T_1^{[*]})^{j-1} T_1^{k+1} y_1 = \underbrace{T_2^* H_1 (T_1^{[*]})^{j-1} T_1^{j-1} T_1}_{=0} T_1^{k-j+1} y_1 = 0$$

and

$$T_2^* H_1 (T_1^{[*]})^{j-1} T_1^k T_2 y_2 = \underbrace{T_2^* H_1 (T_1^{[*]})^{j-1} T_1^{j-1} T_1 T_1^{k-j} T_2 y_2}_{=0} = 0.$$

Moreover,  $T_2^* H_1 (T_1^{[*]})^{j-1} B_{k+1} = 0$ , because for  $i = 2, \dots, k+1$  we have

$$\begin{aligned} T_2^* H_1 (T_1^{[*]})^{j-1} T_1^{k+1-i} T_2 D_i^{(k+1)} &= T_2^* (T_1^*)^{j-1} ((T_1^{[*]})^*)^{k+1-i} H_1 T_2 D_i^{(k+1)} \\ &= \underbrace{(T_2^* H_1 (T_1^{[*]})^{k+1-i} T_1^{k+1-i} T_1 T_1^{j-3-(k-i)} T_2)^*}_{=0} D_i^{(k+1)} = 0 \end{aligned}$$

for the case  $k+1-i < j-1$ .

Furthermore, for the case  $k+1-i = j-1$  we have

$$T_2^* H_1 (T_1^{[*]})^{j-1} T_1^{k+1-i} T_2 D_i^{(k+1)} = \underbrace{T_2^* H_1 (T_1^{[*]})^{j-1} T_1^{j-1} T_2 D_i^{(k+1)}}_{=0} = 0$$

and for the case  $j-1 < k+1-i$

$$T_2^* H_1 (T_1^{[*]})^{j-1} T_1^{k+1-i} T_2 D_i^{(k+1)} = \underbrace{T_2^* H_1 (T_1^{[*]})^{j-1} T_1^{j-1} T_1 T_1^{k+1-i-j} T_2 D_i^{(k+1)}}_{=0} = 0$$

and from the definition of  $B_{k+1}$  in (20) we conclude (ii).

Moreover,  $T_2^* H_1 (T_1^{[*]})^{j-1} C_{k+1} = 0$ , because for  $i = 2, \dots, k+1$  we have

$$\begin{aligned} T_2^* H_1 (T_1^{[*]})^{j-1} T_1^{k+1-i} T_2 \tilde{D}_i^{(k+1)} &= T_2^* (T_1^*)^{j-1} ((T_1^{[*]})^*)^{k+1-i} H_1 T_2 \tilde{D}_i^{(k+1)} \\ &= \underbrace{(T_2^* H_1 (T_1^{[*]})^{k+1-i} T_1^{k+1-i} T_1 T_1^{j-3-(k-i)} T_2)^*}_{=0} \tilde{D}_i^{(k+1)} = 0 \end{aligned}$$

for the case  $k+1-i < j-1$ .

Furthermore, for the case  $k+1-i = j-1$  we have

$$T_2^* H_1 (T_1^{[*]})^{j-1} T_1^{k+1-i} T_2 \tilde{D}_i^{(k+1)} = \underbrace{T_2^* H_1 (T_1^{[*]})^{j-1} T_1^{j-1} T_2 \tilde{D}_i^{(k+1)}}_{=0} = 0$$

and for the case  $j-1 < k+1-i$  we have

$$T_2^* H_1 (T_1^{[*]})^{j-1} T_1^{k+1-i} T_2 \tilde{D}_i^{(k+1)} = \underbrace{T_2^* H_1 (T_1^{[*]})^{j-1} T_1^{j-1} T_1 T_1^{k+1-i-j} T_2 \tilde{D}_i^{(k+1)}}_{=0} = 0$$

and from the definition of  $C_{k+1}$  in (20) we conclude (iv).

Next, we will show that (22) also implies that

$$(v) \quad T_2^* H_1 (T_1^{[*]})^k B_{k+1} y_1 = 0;$$

$$(vi) \quad T_2^* H_1 (T_1^{[*]})^k C_{k+1} y_2 = 0.$$

We show  $T_2^* H_1 (T_1^{[*]})^k B_{k+1} y_1 = 0$ . For  $i = 2, \dots, k+1$  we have

$$\begin{aligned} T_2^* H_1 (T_1^{[*]})^k T_1^{k+1-i} T_2 D_i^{(k+1)} &= T_2^* (T_1^*)^k ((T_1^{[*]})^*)^{k+1-i} H_1 T_2 D_i^{(k+1)} \\ &= \underbrace{(T_2^* H_1 (T_1^{[*]})^{k+1-i} T_1^{k+1-i} T_1 T_1^{k-2-(k-i)} T_2)^*}_{=0} D_i^{(k+1)} = 0. \end{aligned}$$

Moreover,  $T_2^* H_1 (T_1^{[*]})^k C_{k+1} y_2 = 0$ , because for  $i = 2, \dots, k+1$  we have

$$\begin{aligned} T_2^* H_1 (T_1^{[*]})^k T_1^{k+1-i} T_2 \tilde{D}_i^{(k+1)} &= T_2^* (T_1^*)^k ((T_1^{[*]})^*)^{k+1-i} H_1 T_2 \tilde{D}_i^{(k+1)} \\ &= \underbrace{(T_2^* H_1 (T_1^{[*]})^{k+1-i} T_1^{k+1-i} T_1^{i-2-(k-i)} T_2)^*}_{=0} \tilde{D}_i^{(k+1)} = 0. \end{aligned}$$

Thus, by (20), (v) and (vi) hold. We now have all ingredients to prove the equivalence of (a) and (b) under our induction hypothesis (22):

$$\begin{aligned} &(T^{[*]})^{k+1} T^{k+1} \text{ has full domain and (22)} \\ \Leftrightarrow (21) \quad &\begin{cases} T_2^* H_1 (T_1^{[*]})^s ((T_1^{k+1} + B_{k+1}) y_1 + (T_1^k T_2 + C_{k+1}) y_2) = 0, \\ \text{for all } s = 0, \dots, k \text{ and all } y_1, y_2, \text{ and (22)} \end{cases} \\ \Leftrightarrow (i)-(iv) \quad &\begin{cases} T_2^* H_1 (T_1^{[*]})^k ((T_1^{k+1} + B_{k+1}) y_1 + (T_1^k T_2 + C_{k+1}) y_2) = 0, \\ \text{for all } y_1, y_2, \text{ and (22)} \end{cases} \\ \Leftrightarrow (v),(vi) \quad &T_2^* H_1 (T_1^{[*]})^k (T_1^{k+1} y_1 + T_1^k T_2 y_2) = 0, \text{ for all } y_1, y_2, \text{ and (22)} \\ \Leftrightarrow \quad &\begin{cases} T_2^* H_1 (T_1^{[*]})^{i-1} T_1^{i-1} T_1 = 0 \text{ and } T_2^* H_1 (T_1^{[*]})^{i-1} T_1^{i-1} T_2 = 0 \\ \text{for } i = 1, \dots, k+1. \end{cases} \end{aligned}$$

This concludes the proof.  $\square$

Fortunately, it is not necessary to verify that  $(T^{[*]})^k T^k$  has full domain for all  $k \in \mathbb{N}$  in order to show that the  $H$ -hyponormal matrix  $T$  is strongly  $H$ -hyponormal. The following result shows that it is sufficient to check this for all  $k \leq \text{rank } H$ .

**Theorem 4.5.** Let  $T \in \mathbb{C}^{n \times n}$  be a matrix. If  $T$  is strongly  $H$ -hyponormal of degree  $m = \text{rank } H$ , then  $T$  is strongly  $H$ -hyponormal.

**Proof.** It remains to show that  $(T^{[*]})^k T^k$  has full domain for all  $k \in \mathbb{N}$ . We will show this by contradiction. Assuming that  $T$  is not strongly  $H$ -hyponormal, there exists a natural number  $k \geq m$  such that  $T$  is strongly  $H$ -hyponormal of degree  $k$ , but not of degree  $k+1$ . Without loss of generality assume that  $T$  and  $H$  have the forms as in (7). According to Proposition 4.4 being strongly  $H$ -hyponormal of degree  $k$  is then equivalent to

$$T_2^* H_1 (T_1^{[*]})^{i-1} T_1^{i-1} T_1 = 0 \quad \text{and} \quad T_2^* H_1 (T_1^{[*]})^{i-1} T_1^{i-1} T_2 = 0 \quad \text{for } 1 \leq i \leq k.$$

We aim to show that  $(T^{[*]})^{k+1} T^{k+1}$  has full domain in contradiction to the assumption that  $T$  is not strongly  $H$ -hyponormal of degree  $k+1$ . Thus, we have to show

$$T_2^* H_1 (T_1^{[*]})^k T_1^k T_1 = 0 \quad \text{and} \quad T_2^* H_1 (T_1^{[*]})^k T_1^k T_2 = 0. \quad (23)$$

Note that the size of  $T_1$  is  $m \times m$ , as  $m = \text{rank } H$ . Thus, by the Cayley–Hamilton Theorem there exist  $\alpha_0, \dots, \alpha_{m-1} \in \mathbb{C}$  such that

$$(T_1^{[*]})^m = \sum_{i=0}^{m-1} \alpha_i (T_1^{[*]})^i.$$

Hence we see that

$$(T_1^{[*]})^k = (T_1^{[*]})^{k-m} \sum_{i=0}^{m-1} \alpha_i (T_1^{[*]})^i.$$

This gives

$$\begin{aligned} T_2^* H_1 (T_1^{[*]})^k T_1^k T_1 &= \sum_{i=0}^{m-1} \alpha_i T_2^* H_1 (T_1^{[*]})^{k-m+i} T_1^k T_1 \\ &= \sum_{i=0}^{m-1} \alpha_i \underbrace{T_2^* H_1 (T_1^{[*]})^{k-m+i} T_1^{k-m+i} T_1}_{=0} T_1^{m-i} = 0 \end{aligned}$$

and

$$\begin{aligned} T_2^* H_1 (T_1^{[*]})^k T_1^k T_2 &= \sum_{i=0}^{m-1} \alpha_i T_2^* H_1 (T_1^{[*]})^{k-m+i} T_1^k T_2 \\ &= \sum_{i=0}^{m-1} \alpha_i \underbrace{T_2^* H_1 (T_1^{[*]})^{k-m+i} T_1^{k-m+i} T_1}_{=0} T_1^{m-i-1} T_2 = 0 \end{aligned}$$

contradicting the assumption. Thus,  $T$  is strongly  $H$ -hyponormal.  $\square$

## 5. Moore–Penrose $H$ -normal matrices

In this section we will show, how the sets of  $H$ -hyponormal and strongly  $H$ -hyponormal matrices are connected to the set of Moore–Penrose- $H$ -normal matrices. Recall that a matrix  $T \in \mathbb{C}^{n \times n}$  is called Moore–Penrose  $H$ -normal if

$$HTH^\dagger T^* H = T^* HT,$$

where  $H^\dagger$  denotes the Moore–Penrose generalized inverse of  $H$ . The following lemma can be found in [16, Proposition 4.6].

**Lemma 5.1.** *Let  $T$  and  $H$  be given as in (7), then the Moore–Penrose generalized inverse of  $H$  is given by*

$$H^\dagger = \begin{bmatrix} H_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}$$

and the matrix  $T$  is Moore–Penrose  $H$ -normal if and only if

$$\begin{bmatrix} T_1^* H_1 T_1 & T_1^* H_1 T_2 \\ T_2^* H_1 T_1 & T_2^* H_1 T_2 \end{bmatrix} = \begin{bmatrix} H_1 T_1 H_1^{-1} T_1^* H_1 & 0 \\ 0 & 0 \end{bmatrix}. \quad (24)$$

**Remark 5.2.** Note that the equation for the  $(1, 1)$ -block means that  $T_1$  is  $H_1$ -normal with respect to the nondegenerate inner product induced by  $H_1$ .

Moore–Penrose  $H$ -normal matrices were investigated in [11, 12]. Each  $H$ -normal matrix is Moore–Penrose  $H$ -normal, more information is given in the following proposition from [16, Proposition 4.6].

**Proposition 5.3.** *Let  $T \in \mathbb{C}^{n \times n}$  be a matrix. Then the following statements are equivalent:*

- (i)  $T$  is  $H$ -normal, i.e.,  $TT^{[*]} \subseteq T^{[*]}T$ ;
- (ii)  $T$  is Moore–Penrose  $H$ -normal and  $T^{[*]}(T^{[*]})^{[*]} = (T^{[*]})^{[*]}T^{[*]}$ .

In the following example borrowed from [12, Example 6.1], we present a matrix that is Moore–Penrose  $H$ -normal, but not  $H$ -normal.



**Example 5.4.** Let

$$H = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad T = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

then

$$H^\dagger = \frac{1}{16} \begin{bmatrix} 3 & 4 & 5 \\ 4 & 0 & 12 \\ 5 & 12 & 3 \end{bmatrix}$$

and furthermore

$$T^*HT = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad HTH^\dagger T^*H = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Hence  $T$  is Moore–Penrose  $H$ -normal. We easily calculate that

$$\ker H = \operatorname{span} \left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\},$$

which is obviously not  $T$ -invariant. Therefore,  $T$  is not  $H$ -normal.

Hence the  $H$ -normal matrices are indeed a strict subset of the Moore–Penrose  $H$ -normal matrices. The following theorem shows, that in the special case that  $T$  is a matrix and  $T_1$  in (7) is  $H_1$ -normal, the properties Moore–Penrose  $H$ -normal, strongly  $H$ -hyponormal and  $H$ -hyponormal are equivalent.

**Theorem 5.5.** Let  $T \in \mathbb{C}^{n \times n}$  be a matrix and let  $T$  and  $H$  be in the forms as in (7). Then the following assertions are equivalent:

- (i)  $T$  is Moore–Penrose  $H$ -normal;
- (ii)  $T$  is strongly  $H$ -hyponormal and  $T_1$  is  $H_1$ -normal;
- (iii)  $T$  is  $H$ -hyponormal and  $T_1$  is  $H_1$ -normal.

**Proof.** Suppose that (i) holds, i.e.,  $T$  is Moore–Penrose  $H$ -normal. According to Lemma 5.1 this implies

- (a)  $T_1$  is  $H_1$ -normal, i.e.,  $T_1^{[*]}T_1 = T_1T_1^{[*]}$ ;
- (b)  $T_1^*H_1T_2 = 0$ ,  $T_2^*H_1T_1 = 0$  and  $T_2^*H_1T_2 = 0$ .

Therefore,

$$y_1^*H_1(T_1^{[*]}T_1 - T_1T_1^{[*]})y_1 = 0 \quad \text{for all } y_1 \in \mathbb{C}^m,$$

and thus  $T^{[*]}T - TT^{[*]}$  is  $H$ -nonnegative by Proposition 3.3 as  $T_2^*H_1T_2 = 0$ . Furthermore, due to the  $H_1$ -normality of  $T_1$  and the fact that  $T_2^*H_1T_1 = 0$  and  $T_2^*H_1T_2 = 0$ , we obtain

$$T_2^*H_1(T_1^{[*]})^{k-1}T_1^{k-1}T_1 = T_2^*H_1T_1^{k-1}(T_1^{[*]})^{k-1}T_1 = 0 \quad \text{for } k \in \mathbb{N}$$

and

$$T_2^*H_1(T_1^{[*]})^{k-1}T_1^{k-1}T_2 = T_2^*H_1T_1^{k-1}(T_1^{[*]})^{k-1}T_2 = 0 \quad \text{for } k \in \mathbb{N}.$$

This proves that  $(T^{[*]})^kT^k$  has full domain for all  $k \in \mathbb{N}$ , see Proposition 4.4. Hence  $T$  is strongly  $H$ -hyponormal.

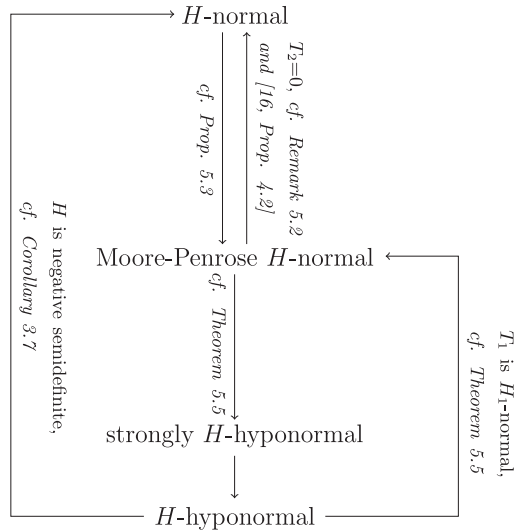
By definition every strongly  $H$ -hyponormal matrix is  $H$ -hyponormal, therefore (ii) implies (iii).

Finally, we will show that (iii) implies (i). Let  $T$  be  $H$ -hyponormal. Then the fact that  $T^{[*]}T$  has full domain implies that  $T_2^*H_1T_1 = 0$  and  $T_2^*H_1T_2 = 0$  (see Proposition 2.6), consequently  $T_1^*H_1T_2 = 0$ .

Together with the assumption of  $T_1$  being  $H_1$ -normal and Lemma 5.1 we find that  $T$  is Moore–Penrose  $H$ -normal.  $\square$

There are strongly  $H$ -hyponormal matrices such that  $T_1$  is not  $H_1$ -normal, see Example 4.2. Hence, according to Theorem 5.5, the Moore–Penrose  $H$ -normal matrices are a strict subset of the strongly  $H$ -hyponormal matrices and, hence, of the  $H$ -hyponormal matrices (see also Example 3.8). The diagram below shows the relation between the different classes of matrices.

**Diagram 5.6**



## 6. Invariant maximal nonpositive subspaces of strongly $H$ -hyponormal matrices

The question under which conditions invariant semidefinite subspaces can be extended to invariant maximal semidefinite subspaces was discussed in [12–14] for  $H$ -normal and  $H$ -hyponormal matrices in the nondegenerate case. The fact that in the case of a degenerate inner product  $\ker H$  remains invariant for an  $H$ -normal matrix was the key to a first generalization of those extension results in [16, Theorems 4.7 and 4.8]. We have seen in Example 4.2 that  $\ker H$  is in general not  $T$ -invariant for a strongly  $H$ -hyponormal matrix  $T$ . The following theorem however describes how to find a  $T$ -invariant subspace that contains  $\ker H$ .

**Theorem 6.1.** *Let  $T \in \mathbb{C}^{n \times n}$  be a strongly  $H$ -hyponormal matrix. Let  $\mathcal{M}$  be the smallest  $T$ -invariant subspace containing the kernel of  $H$ . Then  $\mathcal{M}$  is  $H$ -neutral. In particular, if  $T$  and  $H$  are in the forms (7), then  $\mathcal{M} = \mathcal{M}_0[+] \ker H$ , where  $\mathcal{M}_0$  (canonically identified with a subspace of  $\mathbb{C}^m$ ) is  $H_1$ -neutral and the smallest  $T_1$ -invariant subspace that contains the range of  $T_2$ .*

**Proof.** For the proof, we use an idea similar to the one in [12, Theorem 6.6]. Without loss of generality, we may assume that  $T$  and  $H$  are in the forms (7), where  $T_1 \in \mathbb{C}^{m \times m}$  and  $T_2 \in \mathbb{C}^{m \times (n-m)}$ . Let

$$X = [T_2 \quad T_1 T_2 \cdots T_1^{m-1} T_2] \quad \text{and} \quad \mathcal{M}_0 = \text{Im } X,$$

i.e.,  $\mathcal{M}_0$  is the controllable subspace of the pair  $(T_1, T_2)$ . Therefore,  $\mathcal{M}_0$  is the smallest  $T_1$ -invariant subspace that contains the range of  $T_2$ . In particular, it follows that there exist matrices  $B$  and  $C$  of appropriate dimensions such that

$$T_1 X = X B \quad \text{and} \quad T_2 = X C.$$

Now we identify  $\mathcal{M}_0$  canonically with a subspace of  $\mathbb{C}^n$  and set

$$\widetilde{\mathcal{M}} := \mathcal{M}_0 \dot{+} \ker H = \operatorname{Im} \begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix} \subseteq \mathbb{C}^n.$$

Then  $\widetilde{\mathcal{M}}$  contains  $\ker H$  and by

$$T \begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} T_1 X & T_2 \\ T_3 X & T_4 \end{bmatrix} = \begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} B & C \\ T_3 X & T_4 \end{bmatrix}$$

$\widetilde{\mathcal{M}}$  is  $T$ -invariant. Moreover,  $\widetilde{\mathcal{M}}$  is the smallest  $T$ -invariant subspace containing  $\ker H$ , because

$$T(\ker H) = \operatorname{Im} \begin{bmatrix} T_2 \\ T_4 \end{bmatrix}$$

and thus any  $T$ -invariant subspace containing the kernel of  $H$  must also contain  $\mathcal{M}_0$ . It remains to show that  $\widetilde{\mathcal{M}} = \mathcal{M}$  is  $H$ -neutral, or, equivalently, that  $\mathcal{M}_0$  is  $H_1$ -neutral. Thus, let  $x \in \mathcal{M}_0$ . Then there exist  $x_i \in \mathbb{C}^{(n-m)}$  and  $\alpha_i \in \mathbb{C}$ ,  $i = 1, \dots, m$  such that

$$x = \sum_{i=1}^m \alpha_i T_1^{i-1} T_2 x_i.$$

Then using the fact that  $(T_1^{[*]})^i T_1^i$  has full domain for all  $i$ , or, by Proposition 4.4 equivalently,  $T_2^* H_1 (T_1^{[*]})^{i-1} T_1^{i-1} T_1 = 0$  and  $T_2^* H_1 (T_1^{[*]})^{i-1} T_1^{i-1} T_2 = 0$  for all  $i$ , we obtain that

$$\begin{aligned} x^* H_1 x &= \sum_{i,j=1}^m \alpha_i \bar{\alpha}_j x_j^* T_2^* H_1 (T_1^{[*]})^{j-1} T_1^{i-1} T_2 x_i \\ &= \sum_{i,j=1}^m \alpha_i \bar{\alpha}_j x_j^* \left( T_2^* H_1 (T_1^{[*]})^{i-1} T_1^{j-1} T_2 \right)^* x_i = 0, \end{aligned}$$

hence  $\mathcal{M}_0$  is  $H_1$ -neutral.  $\square$

Finally, we give sufficient conditions for the existence of an invariant maximal  $H$ -nonpositive subspace for strongly  $H$ -hyponormal matrices by giving conditions when the subspace  $\mathcal{M}$  from Theorem 6.1 can be extended to a maximal  $H$ -nonpositive subspace.

**Theorem 6.2.** *Let  $T$  be strongly  $H$ -hyponormal, and let  $\mathcal{M}$  be the smallest  $T$ -invariant subspace containing the kernel of  $H$  which is  $H$ -neutral by Theorem 6.1. Decompose  $\mathcal{M}^{[\perp]}$  as*

$$\mathcal{M}^{[\perp]} = \mathcal{M} \dot{+} \mathcal{M}_{\text{nd}}, \quad (25)$$

for an  $H$ -nondegenerate subspace  $\mathcal{M}_{\text{nd}}$ . Denote by  $\widetilde{T}_{33}$  and  $\widetilde{H}_3$  the compressions of  $T$  and  $H$  to  $\mathcal{M}_{\text{nd}}$ , respectively. Then  $\widetilde{H}_3$  is invertible. Assume that  $\mathcal{M}$  is invariant under  $T^{[*]}$  and that, in addition, one of the three following conditions holds:

- (i)  $\sigma(\widetilde{T}_{33} + \widetilde{T}_{33}^{[*]}) \subset \mathbb{R}$ ;
- (ii)  $\sigma(\widetilde{T}_{33} - \widetilde{T}_{33}^{[*]}) \subset i\mathbb{R}$ ;
- (iii)  $\widetilde{T}_{33}$  is  $H_3$ -normal.

Then  $\mathcal{M}$  can be extended to a maximal  $H$ -nonpositive subspace  $\mathcal{M}_-$  that is invariant under  $T$ .

The conditions (i)–(iii) are independent of the particular choice of a nondegenerate subspace  $\mathcal{M}_{\text{nd}}$  subject to (25).

**Proof.** The proof is divided into three steps. In the first step, we construct an  $H$ -nondegenerated subspace  $\mathcal{M}_3$  such that  $\mathcal{M}^{[\perp]} = \mathcal{M} \dot{+} \mathcal{M}_3$ . In the second step, we show that if one of the conditions (i)–

(iii) for the compression  $T_{33}$  of  $T$  to  $\mathcal{M}_3$  is satisfied, then  $\mathcal{M}$  can be extended to a maximal  $H$ -nonpositive subspace  $\mathcal{M}_-$  that is invariant under  $T$ . In the last step, we show that if one of the conditions (i)–(iii) for the compression of  $T$  to some nondegenerate subspace  $\mathcal{M}_{\text{nd}}$  subject to (25) holds, then this condition is also true for  $T_{33}$ .

*Step (1).* Assume that  $T$  and  $H$  are in the forms (7), with  $T_1, H_1 \in \mathbb{C}^{m \times m}$ . Let  $\mathcal{M}_0$  be as in the proof of Theorem 6.1, i.e.,

$$\mathcal{M} = \mathcal{M}_0[\dot{+}] \ker H.$$

As  $\mathcal{M}$  is  $H$ -neutral, the isotropic part  $\mathcal{M}^\circ$  of  $\mathcal{M}$  equals  $\mathcal{M}$ . By [18, Lemma 3.10] we find a subspace  $\mathcal{M}_{sl}$  skewly linked to  $\mathcal{M}_0$  and a subspace  $\mathcal{M}_3$  with

$$\mathbb{C}^n = ((\mathcal{M}_0 \dot{+} \mathcal{M}_{sl})[\dot{+}] \mathcal{M}_3)[\dot{+}] \ker H, \quad (26)$$

$$\mathcal{M}^{[\perp]} = \mathcal{M}[\dot{+}] \mathcal{M}_3 = \mathcal{M}_0[\dot{+}] \mathcal{M}_3[\dot{+}] \ker H.$$

Note that the space  $\mathcal{M}_3$  is  $H$ -nondegenerate. (This follows from the fact that the space  $\mathcal{M}_0 \dot{+} \mathcal{M}_{sl}$ , canonically identified with a subspace of  $\mathbb{C}^m$ , is  $H_1$ -nondegenerate, and, therefore,  $\mathcal{M}_3$ , also canonically identified with a subspace of  $\mathbb{C}^m$  is the  $H_1$ -orthogonal companion of  $\mathcal{M}_0 \dot{+} \mathcal{M}_{sl}$ . Hence it is  $H_1$ -nondegenerate.) Thus, the restriction  $H_3$  of  $H$  to  $\mathcal{M}_3$  is invertible.

*Step (2).* Assume that one of the conditions (i)–(iii) for the compression  $T_{33}$  of  $T$  to  $\mathcal{M}_3$  is satisfied. As  $\mathcal{M}_0$  (canonically identified with a subspace of  $\mathbb{C}^m$ ) is  $H_1$ -neutral,  $T_1$ -invariant, and contains the range of  $T_2$  we find, with respect to the decomposition (26) that  $H$  and  $T$  have the forms

$$H = \left[ \begin{array}{c|c} H_1 & 0 \\ \hline 0 & 0 \end{array} \right] = \left[ \begin{array}{ccc|c} 0 & I & 0 & 0 \\ I & 0 & 0 & 0 \\ 0 & 0 & H_3 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array} \right],$$

and

$$T = \left[ \begin{array}{c|c} T_1 & T_2 \\ \hline T_3 & T_4 \end{array} \right] = \left[ \begin{array}{ccc|c} T_{11} & T_{12} & T_{13} & T_{14} \\ 0 & T_{22} & T_{23} & 0 \\ 0 & T_{32} & T_{33} & 0 \\ \hline T_{41} & T_{42} & T_{43} & T_{44} \end{array} \right].$$

Then for  $T_1^{[*]}$ , we obtain

$$T_1^{[*]} = \left[ \begin{array}{ccc} T_{22}^* & T_{12}^* & T_{32}^* H_3 \\ 0 & T_{11}^* & 0 \\ H_3^{-1} T_{23}^* & H_3^{-1} T_{13}^* & T_{33}^{[*]} \end{array} \right]$$

and as  $\mathcal{M}$  was assumed to be invariant for  $T^{[*]}$  (recall from [16] that by definition this means “ $x \in \mathcal{M}$  and  $\begin{pmatrix} x \\ y \end{pmatrix} \in T^{[*]} \implies y \in \mathcal{M}$ ”), we obtain that

$$T_1^{[*]} = \left[ \begin{array}{ccc} T_{22}^* & T_{12}^* & T_{32}^* H_3 \\ 0 & T_{11}^* & 0 \\ 0 & H_3^{-1} T_{13}^* & T_{33}^{[*]} \end{array} \right],$$

or, equivalently,  $T_{23} = 0$ . If we decompose  $N := H_1(T_1^{[*]} T_1 - T_1 T_1^{[*]})$  with respect to the decomposition  $(\mathcal{M}_0 \dot{+} \mathcal{M}_{sl})[\dot{+}] \mathcal{M}_3$ , we obtain that the  $(3, 3)$ -block  $N_{33}$  takes the form

$$N_{33} = H_3(T_{33}^{[*]} T_{33} - T_{33} T_{33}^{[*]}).$$

Now we show that  $T_{33}$  is  $H_3$ -hyponormal. Indeed, let  $d$  be the dimension of  $\mathcal{M}_3$ , let  $y_3 \in \mathbb{C}^d$  be arbitrary, and let  $y_1 = (0, 0, y_3)^T \in \mathbb{C}^m$ . Then  $T_2^* H_1 y_1 = 0$  and thus, since  $T$  is  $H$ -hyponormal, we obtain that

$$y_1^* H_1 (T_1^{[*]} T_1 - T_1 T_1^{[*]}) y_1 = y_1^* N y_1 \geq 0.$$

This implies  $y_3^* N_{33} y_3 \geq 0$ , and  $T_{33}$  is  $H_3$ -hyponormal. It is obvious, that  $\{0\}$  is an  $H_3$ -nonpositive,  $T_{33}$ -invariant subspace, with  $\{0\}^{\perp|H_3} = \mathcal{M}_3$ . As  $T_{33}$  was assumed to be  $H_3$ -hyponormal we can use a result from [14, Theorem 6], to obtain that there is a maximal  $H_3$ -nonpositive subspace  $\mathcal{N}_3$  that is invariant under  $T_{33}$  and  $T_{33}^{[*]|H_3}$ . Canonically identifying  $\mathcal{N}_3$  with a subspace of  $\mathbb{C}^n$ , we obtain using the fact that  $T_{23} = 0$  that  $\mathcal{M}_- := \mathcal{M}_0 \dot{+} \mathcal{N}_3 + \ker H$  is a  $T$ -invariant maximal  $H$ -nonpositive subspace containing  $\mathcal{M}$ .

*Step (3).* Next, we show that the conditions (i)–(iii) are independent of the particular choice of a nondegenerate subspace  $\mathcal{M}_{\text{nd}}$ . Indeed, choosing another nondegenerate subspace  $\mathcal{M}_{\text{nd}}$  in

$$\mathcal{M}^{[\perp]} = \text{Im} \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$$

in place of  $\mathcal{M}_3$  amounts to a change of basis in  $\mathbb{C}^n$  given by a matrix of the form

$$S = \begin{bmatrix} I & 0 & S_{13} & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & S_{33} & 0 \\ 0 & 0 & S_{43} & I \end{bmatrix},$$

with an invertible  $S_{33}$ . We have

$$S^{-1} = \begin{bmatrix} I & 0 & -S_{13}S_{33}^{-1} & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & S_{33}^{-1} & 0 \\ 0 & 0 & -S_{43}S_{33}^{-1} & I \end{bmatrix}.$$

Thus, we obtain that with respect to the new decomposition

$$\mathbb{C}^n = ((\mathcal{M}_0 \dot{+} \mathcal{M}_{sl}) \dot{+} \mathcal{M}_{\text{nd}}) \dot{+} \ker H$$

and the new basis, the matrices of interest take the form

$$\tilde{T} = S^{-1}TS = \begin{bmatrix} * & * & * & * \\ 0 & * & 0 & 0 \\ 0 & * & S_{33}^{-1}T_{33}S_{33} & 0 \\ * & * & * & * \end{bmatrix}$$

and

$$\tilde{H} = S^*HS = \begin{bmatrix} 0 & I & 0 & 0 \\ I & 0 & S_{13} & 0 \\ 0 & S_{13}^* & S_{33}^*H_3S_{33} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus the compressions  $\tilde{T}_{33}$  and  $\tilde{H}_3$  of  $\tilde{T}_1$ , respectively,  $\tilde{H}_1$  to  $\mathcal{M}_{\text{nd}}$  are

$$\tilde{T}_{33} = S_{33}^{-1}T_{33}S_{33}, \quad \tilde{H}_3 = S_{33}^*H_3S_{33}.$$

Clearly it follows from this that if each of the three conditions (i)–(iii) holds for  $\tilde{T}_{33} = S_{33}^{-1}T_{33}S_{33}$  and  $\tilde{H}_3 = S_{33}^*H_3S_{33}$ , then it holds also for  $T_{33}$  and  $H_3$ . In particular, the conditions (i)–(iii) are independent of the choice of  $\mathcal{M}_{\text{nd}}$ .  $\square$

Theorem 6.2 generalizes [12, Theorem 7]. Note that already in the nondegenerate case additional assumptions were necessary to guarantee existence of invariant maximal nonpositive subspaces and it was shown in [12] that these assumptions were essential.

## 7. Conclusions

We have extended the notion of  $H$ -hyponormality to the case of degenerate indefinite inner products. The straightforward approach to define  $H$ -hyponormality, i.e., calling a matrix  $T$   $H$ -hyponormal if

$$H(T^{[*]}T - TT^{[*]}) \geq 0,$$

turned out not to be satisfactory. Therefore, we developed the new concept of strong hyponormality. The set of strongly hyponormal matrices has the following three useful properties:

- (i) it contains the sets of  $H$ -normal and Moore–Penrose  $H$ -normal matrices;
- (ii) it equals the set of  $H$ -normal matrices if  $H$  is negative semidefinite;
- (iii) any strongly  $H$ -normal matrix has an invariant  $H$ -neutral subspace containing  $\ker H$ .

In particular, we have shown how the latter property can be used to generalize existence results for invariant maximal nonpositive subspaces from the case of nondegenerate indefinite inner products to degenerate ones.

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